MATH/STAT 394, Homework 3

Jess Kunke

Due Mon 11 July 2022

Remember to refer to the syllabus for homework instructions and guidelines. Note that not all parts of these problems are equally hard or time-consuming.

Required exercises

Exercise 1. Let X have the following probability mass function:

x	1	2	3	4
$P_X(x)$	2/5	1/5	1/5	1/5

1. Calculate $P(X \geq 2)$ and $P(X > 2)$.
2. Calculate $P(X \leq 3 \mid X \geq 2)$.
3. Specify the cumulative distribution function $F_X(x)$ for each possible value x of X.

Exercise 2. Consider a random variable Z with cdf

$$F(z) = \begin{cases} 0 & z < 1, \\ \frac{1}{8} & 1 \leq z < 2, \\ \frac{3}{8} & 2 \leq z < 5, \\ \frac{7}{8} & 5 \leq z < 8, \\ 1 & z \geq 8. \end{cases}$$

1. Does this random variable have a pmf $p_Z(z)$ or a pdf $f_Z(z)$? Specify that function for Z; in other words, don't just define what a pmf or pdf in general is, but provide the specific form of that function for this random variable Z.
2. Find $P(Z = 2)$ and $P(Z = 3)$.
3. Find $P(Z \geq 3)$.

Exercise 3. Let $\lambda > 0$ and let X be a continuous random variable with density

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

1. What is the name of this distribution?
2. Compute $P(2 < X < 3)$ and $P(X = 2)$ for $\lambda = 3$.
3. Compute $P(X > t)$, $P(X > s + t | X > t)$ in terms of $s, t \geq 0$, $\lambda > 0$.

4. Usually, the probability that John waits less than 5 min at the bus stop before it arrives is 1/4. Given that he has already been waiting 10 min, what is the probability that he wait at least 5 more minutes? Model the time that John waits for the bus as an exponential RV.

Exercise 4. A pdf is defined as

$$f(x) = \begin{cases} C(x + \frac{3}{2}), & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

1. Find the value of C.
2. Find the expectation and variance of X.
3. Find the expectation of the random variable $Z = \frac{X}{2X+3}$.

Exercise 5. There are six closed boxes on the table. Three of the boxes have prizes inside while the other three are empty. You open the boxes one at a time randomly until you find a price. Let X be the number of boxes you open.

1. Find the probability mass function of X.
2. Find $\mathrm{Var}(X) = \mathrm{E}[X^2] - \mathrm{E}[X]^2$.
3. Suppose the prize inside each of the three boxes is $100, but each empty box you open costs you $100. What is your expected gain or loss in this game?

Extra credit

Exercise 6. A stick of length ℓ is broken at a uniformly chosen random location. We denote by X the length of the smaller piece.

1. Find the cumulative distribution function of X.

   Hint: Express X as the minimum of two random variables.
2. Find the probability density function of X.